

First-order phase transition with a logarithmic singularity in a model with absorbing states

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(Received 18 August 2000; published 20 December 2000)

Recently, Lipowski [Phys. Rev. E **62**, 4401 (2000)] investigated a stochastic lattice model which exhibits a discontinuous transition from an active phase into infinitely many absorbing states. Since the transition is accompanied by an apparent power-law singularity, it was conjectured that the model may combine features of first- and second-order phase transitions. In the present work it is shown that this singularity emerges as an artifact of the definition of the model in terms of products. Instead of a power law, we find a logarithmic singularity at the transition. Moreover, we generalize the model in such a way that the second-order phase transition becomes accessible. As expected, this transition belongs to the universality class of directed percolation. [S1063-651X(00)13512-3]

DOI: 10.1103/PhysRevE.63.016109

PACS number(s): 05.70.Ln, 64.60.Ak, 64.60.Ht

I. INTRODUCTION

In nonequilibrium statistical physics the study of phase transitions continues to attract considerable attention [1]. In this context, continuous phase transitions into absorbing states have been of particular interest [2]. It is believed that absorbing-state transitions can be categorized into universality classes, the most prominent ones being directed percolation [3] and the so-called parity-conserving universality class [4]. On the other hand, various nonequilibrium models are known to exhibit a discontinuous phase transition [5]. Especially in one spatial dimension, first-order transitions require a very robust mechanism in order to stabilize the ordered phases. As suggested in Ref. [6], first-order transitions in one dimension should be impossible under certain generic conditions if one of the ordered phases fluctuates.

Recently, Lipowski and co-workers introduced a model with infinitely many absorbing states which exhibits a first-order transition from a fluctuating active state into an absorbing phase [7,8]. Remarkably, this transition takes place even in one spatial dimension. Unlike previously investigated models, the dynamic rules involve *products* of real-valued local variables. Surprisingly, the first-order transition is accompanied by an apparent power-law singularity of the stationary particle density, suggesting that the model may combine features of continuous and discontinuous phase transitions. This observation collides with the commonly accepted belief that there are no power-law singularities in first-order phase transitions. The aim of this work is to study the origin of this unusual type of singularity in more detail.

The model considered in Ref. [8] is defined as follows. Each site i of a given lattice is connected to n neighboring sites $j \in \langle i \rangle$. Each bond carries a real-valued variable $w_{ij} = w_{ji} \in (-1/2, +1/2)$. The set of all bond variables specifies the state of the system. A site is considered to be active if the product of all adjacent bond variables is smaller than a certain control parameter r :

$$\prod_{j \in \langle i \rangle} w_{ij} < r. \quad (1)$$

Initially all bond variables are uniformly distributed between

$-1/2$ and $+1/2$. The model evolves by random sequential updates according to the following dynamic rules. For each update attempt a site is randomly selected. If it is active, all adjacent bond variables are replaced by new random numbers distributed between $-1/2$ and $+1/2$. Lipowski considered the case of $n=4$ neighbors using a one-dimensional triangular ladder and a two-dimensional square lattice (see Fig. 1).

Due to the use of real-valued local variables, the model has infinitely many absorbing states. Moreover, the dynamic rules are invariant under certain gauge transformations. For example, we may invert the sign of all bond variables along a closed contour without changing the pattern of activity in a given configuration. The implications of this type of gauge invariance are not yet fully understood.

In order to understand the existence of a phase transition, let us consider two extremal situations. On the one hand, for $r > 2^{-n}$ it is obvious that all sites are active during the entire temporal evolution. On the other hand, for $r < 0$ there is a finite probability to generate bond variables with $|w_{i,j}| < 2^{n-1}|r|$. This means that certain pairs of sites i and j remain inactive forever, irrespective of the values of the other bond variables. Thus, the process continuously “switches off” certain pairs of sites, and therefore approaches an absorbing configuration within an exponentially short time.

Interestingly, for $r=0$ the model is still in the active phase with a nonvanishing stationary density of active sites $\rho_0 > 0$ (see Fig. 2). Thus the spreading process undergoes a *discontinuous* phase transition at $r=0$. Even more surpris-

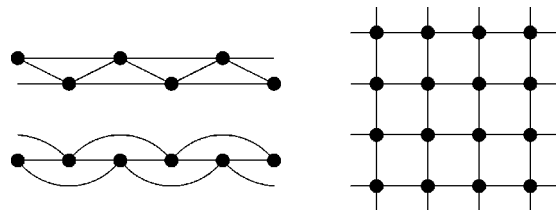


FIG. 1. Lattice geometries used in Ref. [8] Left: one-dimensional triangular ladder, interpreted as a linear chain with next-to-nearest-neighbor interactions. Right: two-dimensional square lattice.

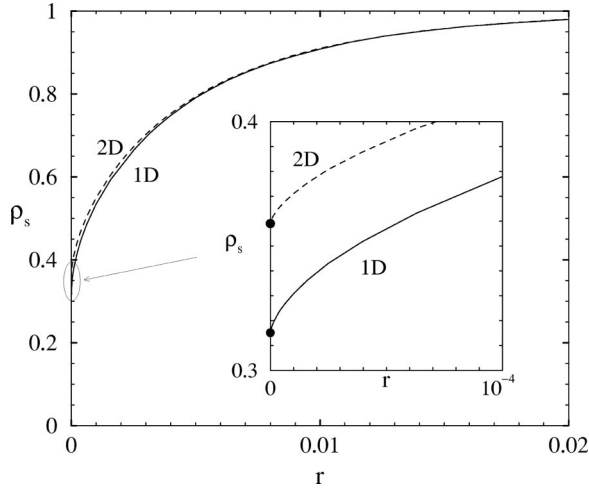


FIG. 2. Stationary density ρ_s as a function of the parameter r in one (1D) and two (2D) spatial dimensions. The inset zooms the region where the curves terminate. The terminal points at $\rho(0) = \rho_0$ are marked by bold dots.

ingly, the stationary density $\rho_s(r)$ does not decrease linearly versus ρ_0 as $r \rightarrow 0$; instead the slope of the curve seems to diverge. This observation led Lipowski to the conjecture that the model may combine features of discontinuous and continuous transitions, calling for a power-law behavior of the forms

$$\begin{aligned} \rho_s(r) &\approx \rho_0 + ar^\beta \quad \text{for } r \geq 0, \\ \rho_s(r) &= 0 \quad \text{for } r < 0, \end{aligned} \quad (2)$$

where a is a certain factor, and β is the critical exponent associated with the order parameter ρ . Performing numerical simulations Lipowski found the estimates $\rho_0 = 0.314827$ and $\beta = 0.66(3)$ in one dimension, and $\rho_0 = 0.358$ and $\beta = 0.58(1)$ in two dimensions, respectively. Moreover, he observed that the dynamical critical exponent $z = \nu_\perp / \nu_\parallel \approx 0.2$ is very small. In the present paper we propose a different explanation of the diverging slope in Fig. 2 based on the following arguments:

(1) The singularity of the slope in Fig. 2 emerges as an artifact of the definition of activity in terms of a product. While the bond variables are uniformly distributed, the probability distribution of the product diverges for $r \rightarrow 0$, leading to a singularity of the slope. An explicit formula is derived, showing that the slope diverges logarithmically as $r \rightarrow 0$. In particular, there is no power law of the form of Eq. (2).

(2) Redefining the control parameter, the model displays a conventional first-order phase transition without a diverging slope. Moreover, there is no diverging length scale at the transition.

(3) Since for $r < 0$ the system is immediately driven toward one of the absorbing states, the continuous transition may be thought of as being hidden in the inaccessible region $r < 0$. In order to support this point of view, we generalize the model in such a way that the continuous transition is shifted to the accessible region $r \geq 0$. As expected, the transition belongs to the universality class of directed percolation.

II. LOGARITHMIC SINGULARITY

In this section we demonstrate that the singularity of the slope in Fig. 2 is a consequence of the multiplicative definition of activity in Eq. (1). To this end we consider the *reactivation probability* $W(r)$ that a site remains active after an update. In a spreading process, the reactivation probability provides a good measure of the effective spreading rate by which nearest neighbors will be activated. In most spreading processes near the transition, the reactivation probability varies to lowest order linearly with the control parameter of the model. Therefore, a power law of the form $\rho_s \sim [W(r) - W(r_c)]^\beta$ immediately implies the same power law in terms of the control parameter $\rho_s \sim (r - r_c)^\beta$. In the present model, however, $W(r)$ is a *nonlinear* function at the transition. Therefore, it does matter whether r or $W(r)$ is used as control parameter.

Our explanation relies on the assumption that $W(r)$ is the “true” control parameter of the model. In terms of $W(r)$ the model exhibits a regular first-order phase transition without singularity, i.e., the density varies linearly with $W(r) - W(r_c)$. The apparent singularity in terms of r originates solely in the nonanalytic behavior of the function $W(r)$ in the limit $r \rightarrow 0$.

In order to compute $W(r)$ let us consider n independent random numbers $z_1, z_2, \dots, z_n$ drawn from a flat distribution between $-1/2$ and $+1/2$. Clearly, the probability $P^{(1)}(z)dz$ to find one of these random numbers between z and $z + dz$ is constant for $|z| \leq 1/2$. However, the product $z = \prod_{i=1}^n z_i$ of the random numbers is not uniformly distributed. The probability distribution $P^{(n)}(z)$ can be computed iteratively by

$$\begin{aligned} P^{(k+1)}(z) &= \int_{-1/2}^{+1/2} dz' \int_{-1/2}^{+1/2} dz'' P^{(k)}(z') \delta(z - z' z'') \\ &= \int_{-1/2}^{+1/2} dz' \frac{1}{z'} P^{(k)}(z/z') \Theta(z' - |z| 2^k) \\ &= 2 \int_{-2^k |z|}^{+1/2} dz' \frac{1}{z'} P^{(k)}(|z|/z'), \end{aligned} \quad (3)$$

with $P^{(1)}(z) = 1$, leading to the exact result

$$P^{(n)}(z) = \frac{(-2)^{n-1}}{(n-1)!} (\ln_e 2^n |z|)^{(n-1)}. \quad (4)$$

Thus the probability $W(r)$ to reactivate an updated site is given by

$$\begin{aligned} W(r) &= \int_{-1/16}^r dz P^{(4)}(z) = \frac{1}{2} + r \left(8 - 8(\ln_e 16|r|) \right. \\ &\quad \left. + 4(\ln_e 16|r|)^2 - \frac{4}{3}(\ln_e 16|r|)^3 \right). \end{aligned} \quad (5)$$

For small values of r , we therefore expect the stationary density to be given by

TABLE I. Numerical estimates.

	$d=1$	$d=2$
Number of sites	10^6	1000^2
Simulation time	$10^4 \dots 10^6$	$10^4 \dots 10^6$
Stationary density ρ_0	0.31512(3)	0.35905(2)
Fit parameter A	1.4(1)	1.1(1)

$$\rho_s(r) \approx \rho_0 + A \left(W(r) - \frac{1}{2} \right) \quad \text{for } r \geq 0, \quad (6)$$

$$\rho_s(r) = 0 \quad \text{for } r < 0,$$

where A is a fit parameter. This formula replaces the power law in Eq. (2), implying that the slope of $\rho_s(r)$ diverges *logarithmically* as $-(\ln_e 16|r|)^3$ for $r \rightarrow 0$ in any dimension. This explains why Lipowski's results in one and two dimensions are so similar.

In order to support the validity of Eq. (6), we performed Monte Carlo simulations (see Table I). Our estimate for the stationary density ρ_0 in one dimension deviates slightly from the value quoted in Ref. [8]. This deviation may be explained as follows. On the one hand, the random number generator plays a crucial role. Since most algorithms internally generate an integer random number, the output is often quantized in steps of about 10^{-8} , leading to wrong results if r is very small. On the other hand, the machine precision itself limits the accuracy. This problem can be avoided by storing $\log(|w_{ij}|)$ and $\text{sgn}(w_{ij})$ instead of $w_{i,j}$ and turning the product in Eq. (1) into a sum of logarithms. Taking these technical subtleties into account, we obtain a different estimate.

As shown in Fig. 3, the results for the stationary density in one and two dimensions are very similar. Moreover, for small values of r the curvature of the lines is in fair agreement with the theoretical prediction of Eq. (6). In any case, the possibility of a power-law singularity can be ruled out. As expected, there is no singularity if ρ_s is plotted against $W(r) - W(0)$ (see Fig. 4).

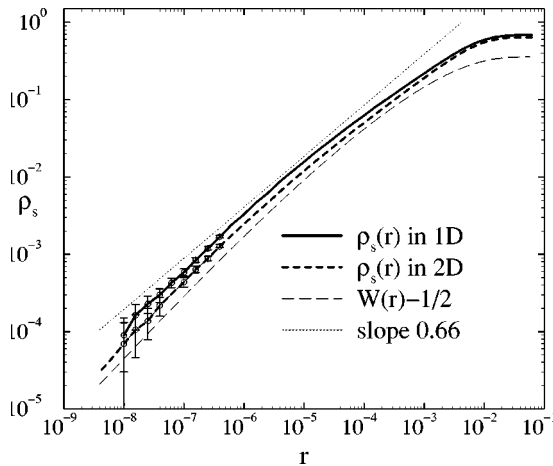


FIG. 3. Log-log plot of the stationary density $\rho_s(r)$ as a function of the parameter r , compared to the (vertically shifted) function $W(r) - 1/2$. The dotted straight line visualizes the failure of the power-law conjecture proposed in Ref. [8].

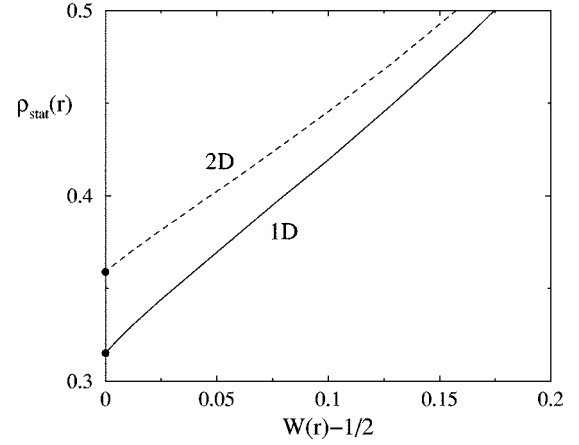


FIG. 4. Stationary density ρ_s as a function of $W(r) - 1/2$. As can be seen, there is no singularity near the transition.

III. CONTINUOUS TRANSITION IN A GENERALIZED MODEL

As outlined in Sec. I, the first-order phase transition at $r = 0$ is induced by frozen pairs of inactive sites. Since for any negative value of r there is a finite probability to generate such pairs during the temporal evolution, the system is driven into one of the absorbing states within an exponentially short time. Roughly speaking, the spreading process is switched off as soon as $r < 0$. Thus, the continuous transition, which is expected to exist in ordinary spreading processes, may be thought of as being hidden in the inaccessible region $r < 0$.

In order to support this point of view, we generalize the model in such a way that the continuous transition is shifted to the accessible region $r \geq 0$. This can be done by introducing two control parameters r_1 and r_2 , and considering a site to be active if the product of adjacent bond variables lies in the interval $(-r_1, r_2)$. Clearly, this model includes the original one as a special case. Moreover, the phase diagram is symmetric under exchange $r_1 \leftrightarrow r_2$. In order to avoid frozen pairs of sites, we will assume that both parameters are positive. Obviously, for very small values of r_1 and r_2 the

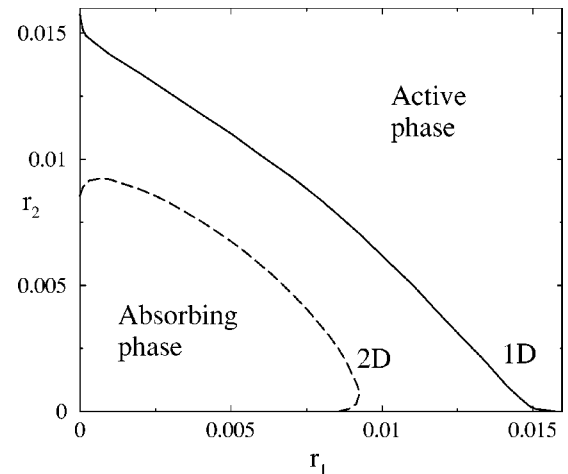


FIG. 5. Phase diagram of the generalized model. The solid (dashed) line indicates the phase transition line in one (two) dimension(s).

spreading probability is so small that the model will be in the absorbing phase. Increasing r_1 and r_2 , we observe a continuous transition from the absorbing to the active phase, as shown in Fig. 5.

We verified that the critical behavior along the entire transition line belongs to the universality class of directed percolation. Note that the Z_2 symmetry along the diagonal $r_1 = r_2$ does not lead a different type of transition since it is not a symmetry of the order parameter.

IV. CONCLUSIONS

The common feature of the models introduced by Lipowski and co-workers is the use of products of real-valued local variables in the definition of the dynamic rules. The

models display very interesting phenomena which are usually not observed in models with linear local rules. As we showed in the present paper, the use of nonlinear rules is crucial, since nonlinear functions of random numbers may not be uniformly distributed. A special situation emerges if the distribution exhibits a singularity. In this case a tiny change of the control parameter may lead to a dramatic variation of the order parameter. For the model investigated in Ref. [8], such a singularity is responsible for the diverging slope in Fig. 2. Thus, in contrast to a previous conjecture, the model does not combine features of first- and second-order phase transitions. In particular, the model is not critical, and the correlation length remains finite as $r \rightarrow 0$. We expect that similar phenomena may emerge whenever the dynamic rules are defined in terms of nonlinear functions of real-valued random variables.

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